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WEIERSTRASS NORMAL FORMS AND INVARIANTS OF ELLIPTIC SURFACES

BY

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ABSTRACT. Let $\pi\colon S \to B$ be an elliptic surface with a section $\sigma\colon B \to S$. Let $L^{-1} \to B$ be the normal bundle of $\sigma(B)$ in S, and let $W = P(L^{\otimes 2} \oplus L^{\otimes 3} \oplus 1)$ be a $\mathbf{P}^2$-bundle over B. Let S^* be the surface obtained from S by contracting those components of fibres of S which do not intersect $\sigma(B)$. Then S^* may be imbedded in W and defined by a "Weierstrass equation":

$$y^2 z = x^3 - g_2 x z^2 - g_3 z^3$$

where $g_2 \in H^0(B, \mathcal{O}(L^{\otimes 4}))$ and $g_3 \in H^0(B, \mathcal{O}(L^{\otimes 6}))$. The only singularities (if any) of S^* are rational double points. The triples (L, g_2, g_3) form a set of invariants for elliptic surfaces with sections, and a complete set of invariants is given by $\{(L, g_2, g_3)\}/G$ where $G \cong \mathbf{C}^* \times \mathrm{Aut}\,(B)$.

An elliptic surface is a morphism $\pi\colon S \to B$ where S is a compact complex analytic surface, B is a compact Riemann surface, and such that for all but finitely many points $t \in B$, $C_t = \pi^{-1}(t)$ is a nonsingular elliptic curve in S. Throughout this paper we will assume the existence of a section $\sigma\colon B \to S$ ($\pi \cdot \sigma = \mathrm{id}_B$). In this case, it follows that S is algebraic [3].

$\pi\colon S \to B$ will be called a minimal elliptic surface if no fibre of S contains an exceptional curve of the first kind. It is possible for $\pi\colon S \to B$ to be a minimal elliptic surface while S is not a minimal surface (rational elliptic surface). If $\pi\colon S \to B$ and $\phi\colon F \to B$ are elliptic surfaces with sections $\sigma\colon B \to S$, $\tau\colon B \to F$, then a birational mapping is a biholomorphic map $f\colon \pi^{-1}(B') \to \phi^{-1}(B')$ where $B' \subset B$ is a Zariski open set, satisfying: $\phi \cdot f = \pi$ and $f \cdot \sigma = \tau$. Then we have the following

THEOREM. *If F is a minimal elliptic surface, then f extends to a holomorphic mapping $\hat{f}\colon S \to F$* [3].

It is not hard to prove that if $\pi\colon S \to B$ is any elliptic surface (not necessarily containing a section), then the exceptional curves lying in any fibre of S are disjoint. It follows that there exists a unique minimal model in any birational class of elliptic surfaces [3].

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Now let π: $S \to B$ be a minimal elliptic surface. If $K(S)$ and $K(B)$ denote the function fields of S and of B respectively, then $K(S)$, as an algebraic function field in one variable over $K(B)$, is of genus 1, and contains a rational point corresponding to the section. It follows that S is birationally equivalent to a (possibly singular) elliptic surface π': $S' \to B$ given by a Weierstrass equation. That is, $S' \subset B \times \mathbf{P}^2$ is defined by an equation of the form:

$$y^2 z = x^3 - g_2 x z^2 - g_3 z^3$$

where g_2, g_3 in $K(B)$ are uniquely determined up to the transformation: $(g_2, g_3) \to (h^4 g_2, h^6 g_3)$, $h \in K(B)$.

In this paper, we wish to describe a (possibly singular) elliptic surface: $\pi^*: S^* \to B$ closely related to the Weierstrass surface S', such that S^* satisfies:

(i) The only singularities of S^* are rational double points;

(ii) S is the minimal resolution of S^*.

Abstractly, S^* is obtained from S by contracting those curves in the singular fibres of S which do not meet the section. We wish to describe a Weierstrass-type equation for S^*.

Let A be the unique divisor on B such that if:

$$\operatorname{div}(g_2) + 4A = \sum_{P \in B} n_P P,$$

$$\operatorname{div}(g_3) + 6A = \sum_{P \in B} m_P P$$

then

(i) $n_P \geqslant 0$, $m_P \geqslant 0$ for all $P \in B$,

(ii) $\min(3n_P, 2m_P) < 12$ for all $P \in B$,

i.e., either $n_P < 4$ or $m_P < 6$. If (g_2, g_3) is replaced by $(h^4 g_2, h^6 g_3)$, then A is replaced by $A - \operatorname{div}(h)$. Thus the divisor class of A is uniquely determined by the elliptic surface S. Let $L = [A]$ be the line bundle of A, and let (l_{ij}) be a system of transition functions for L with respect to some covering $\{U_i\}$ of B. The meromorphic functions g_2, g_3 determine sections:

$$g_2^* \in H^0(B, \mathcal{O}_B(4L)), \qquad g_3^* \in H^0(B, \mathcal{O}_B(6L)).$$

(g_2^*, g_3^*) are determined by S up to the transformation: $(g_2^*, g_3^*) \to (\lambda^4 g_2^*, \lambda^6 g_3^*)$, $\lambda \in \mathbf{C}^*$. g_2^* and g_3^* may be described by systems of holomorphic functions (g_{2i}^*) and (g_{3i}^*) defined on U_i satisfying

$$g_{2i}^* = l_{ij}^4 g_{2j}^*, \qquad g_{3i}^* = l_{ij}^6 g_{3j}^*,$$

on $U_i \cap U_j$.

Let $W = 2L \oplus 3L \oplus 1$. Let $S^* \subset P(W)$ be such that S^* is defined over each piece U_i by the equation:

$$y_i^2 z_i = x_i^3 - g_{2i}^* x_i z_i^2 - g_{3i}^* z_i^3$$

where $(x_i : y_i : z_i)$ are fibre homogeneous coordinates for $P(W)$ over U_i satisfying

$$x_i = l_{ij}^2 x_j, \quad y_i = l_{ij}^3 y_j, \quad z_i = z_j,$$

over $U_i \cap U_j$.

If $B' = B - \text{supp}(A)$, then it is clear that $S^*|_{B'} \cong S'|_{B'}$ and therefore S^* is birationally equivalent to S. We will prove that the only singularities of S^* are rational double points. Notice that we have:

$$\min(3 \operatorname{ord}_P(g_2^*), 2 \operatorname{ord}_P(g_3^*)) < 12$$

at every point $P \in B$.

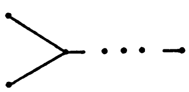
LEMMA 1. *Consider the isolated singularity*

$$y^2 = x^3 - \alpha t^n x - \beta t^m, \qquad n > 0, m > 1,$$

in $\mathbf{C}^3_{(x,y,t)}$ *where* $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\alpha(0) \neq 0$, $\beta(0) \neq 0$, *and where we assume that* $\Delta = 4\alpha^3 t^{3n} - 27\beta^2 t^{2m}$ *is not identically zero. Then the above singularity at the origin is a rational double point if and only if* $\min(3n, 2m) < 12$.

PROOF. We can resolve the singularity explicitly as in [2, p. 81]. We give a table here which describes the graph of the minimal resolution in case $n < 4$ or $m < 6$. We follow the notation of [1], namely:

A_n : (n vertices)

D_n :  (n vertices)

E_n :  (n vertices, $n = 6, 7,$ or 8)

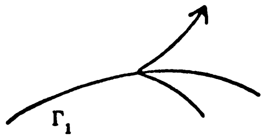
here each vertex represents a nonsingular rational curve with self-intersection -2.

Resolution of $y^2 = x^3 - \alpha t_n x - \beta t^m$

n	m	
$\geqslant 3$	4	E_6
2	$\geqslant 3$	D_4
3	$\geqslant 2$	D_4
3	$\geqslant 5$	E_7
$\geqslant 4$	5	E_8
$\geqslant 2$	2	A_2
1	$\geqslant 2$	A_1

As an illustration, we will carry out the resolution of $y^2 = x^3 - \alpha t^n x - \beta t^5$, $n \geqslant 4$. After blowing up the origin in the (x, t) plane we get the diagram:

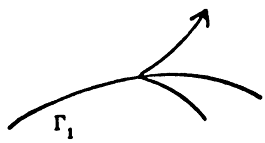
(1)



$\Gamma_1^2 = -1$

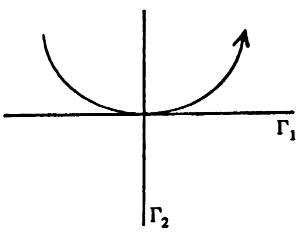
$m_1 = 3$

where Γ_1 is the exceptional curve,



is the proper transform of "$x^3 - \alpha t^n x - \beta t^5 = 0$" which has a simple cusp, and m_1 is the multiplicity of Γ_1 as a component of the divisor of $x^3 - \alpha t^n x - \beta t^5$. After blowing up the cusp, we get the diagram:

(2)

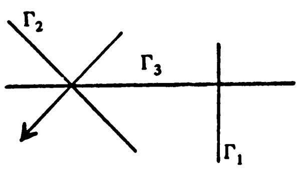


$\Gamma_1^2 = -2 \quad m_1 = 3$

$\Gamma_2^2 = -1 \quad m_2 = 5$

Here Γ_1 represents the proper transform of the Γ_1 of diagram (1). Now, blow up the triple intersection:

(3)



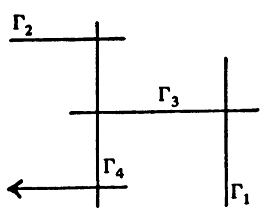
$\Gamma_1^2 = -3 \quad m_1 = 3$

$\Gamma_2^2 = -2 \quad m_2 = 5$

$\Gamma_3^2 = -1 \quad m_3 = 9$

Again blow up the triple intersection:

(4)



$\Gamma_1^2 = -3 \quad m_1 = 3$

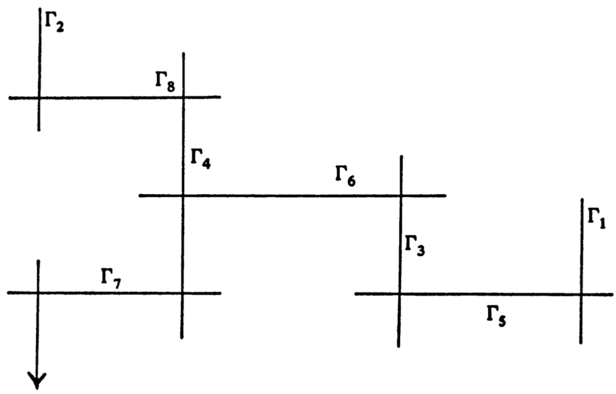
$\Gamma_2^2 = -3 \quad m_2 = 5$

$\Gamma_3^2 = -2 \quad m_3 = 9$

$\Gamma_4^2 = -1 \quad m_4 = 15$

Now blow up each double point. This is to insure that the curves Γ_i with m_i odd are disjoint from one another and from ↗.

(5)



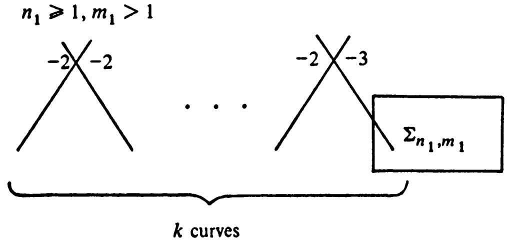
$$\Gamma_1^2 = \Gamma_2^2 = \Gamma_3^2 = \Gamma_4^2 = -4$$
$$\Gamma_5^2 = \Gamma_6^2 = \Gamma_7^2 = \Gamma_8^2 = -1$$
$$m_1, m_2, m_3, m_4 \equiv 1\ (2)$$
$$m_5, m_6, m_7, m_8 \equiv 0\ (2)$$

The resolution is then the double covering of diagram (5) ramified over the curves Γ_1, Γ_2, Γ_3, Γ_4, and $\nearrow$. This is clearly the graph E_8.

By tautness of rational double points [1], it follows that $y^2 = x^3 - \alpha t^n x - \beta t^m$ is a rational double point if $\min(3n, 2m) < 12$.

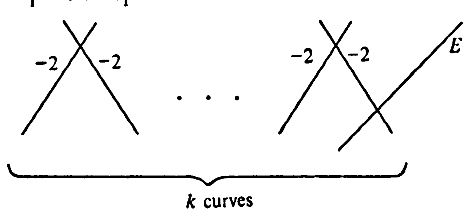
Assume now that $n = 4k + n_1$, $m = 6k + m_1$ where $k > 0$, $\min(3n_1, 2m_1) < 12$. Then the graph of the resolution of $y^2 = x^3 - t^n \alpha x - t^m$ is one of the following:

$n_1 \geqslant 1, m_1 > 1$

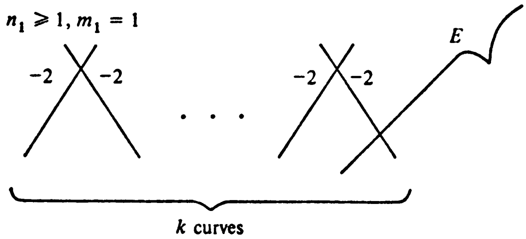


where Σ_{n_1,m_1} is taken from the table above;

$n_1 = 0$ or $m_1 = 0$



where E is an elliptic curve with $E^2 = -1$;



where E is a rational curve with one cusp, and $E^2 = -1$.

In any case, the singularity is not a rational double point. This completes the proof of Lemma 1.

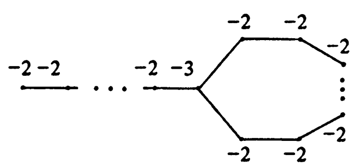
We now consider the singularities of S^* corresponding to poles of j. Thus we consider the surface defined in $\mathbf{C}^2_{(x,y)} \times \{|t| < \varepsilon\}$ by $y^2 = x^3 - \alpha(t)x - \beta(t)$, with discriminant $\Delta = 4\alpha^3 - 27\beta^2$ and invariant, $j = 4\alpha^3/\Delta$. We now assume that j has a pole of order $r > 0$ at $t = 0$. If we set $\alpha(t) = t^n\alpha_1(t)$, $\beta(t) = t^m\beta_1(t)$ with $\alpha_1(0) \neq 0$, and $\beta_1(0) \neq 0$, then we must have $(n, m) = (2k, 3k)$ for some $k \geqslant 0$. Notice that after an analytic change of coordinates, the above equation may be transformed to

$$y^2 = (x - t^k)(x^2 - t^{r+2k}\gamma), \qquad \gamma = \gamma(t), \gamma(0) \neq 0.$$

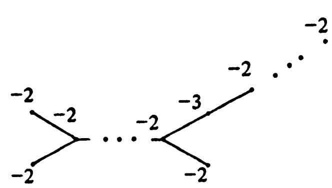
Here again, this singularity may be resolved explicitly by the methods of [2]. We get the following result:

LEMMA 2. *The singularity* $y^2 = (x - t^k)(x^2 - t^{r+2k}\gamma)$ *is not a rational double point if* $k > 1$. *It is a rational double point of type* D_{r+4} *if* $k = 1$, *and of type* A_{r-1} *if* $k = 0, r > 1$.

To be more specific, if $k > 1$ is even, then the graph of the resolution is:



If $k > 1$ is odd, the graph is:



If $k = 0$, the corresponding fibre is of type I_r, while if $k = 1$, the fibre is of type I_r^*.

It is clear from Lemmas 1 and 2 and our construction of the elliptic surface S^* that the only singularities of S^* are rational double points.

To each point $a \in B$, let C_a^* be the fibre of S^* over a. If S is the minimal resolution of S^*, then the fibre of S over a is of the f^rm:

$$C_a = C_{a0} + \sum_{j \geqslant 1} n_j C_{aj}$$

where C_{a0} is the proper transform of C_a^* and where $\bigcup_{j \geqslant 1} C_{aj}$ (if nonempty) form the minimal resolution of a rational double point. Thus we have $C_{aj}^2 = -2$, $K \cdot C_{aj} = 0$ $(j \geqslant 1)$. Since C_a is a fibre of an elliptic surface, $K \cdot C_a = 0$. It follows that $K \cdot C_{a0} = 0$. Thus C_{a0} is not an exceptional curve of the first kind. We may conclude that S is a minimal elliptic surface. We sum up our results.

THEOREM 1. *Let π: $S \to B$ be a minimal elliptic surface which admits a section. Then there exists a line bundle L on B and sections $g_2 \in H^0(B, \mathcal{O}_B(4L))$, $g_3 \in H^0(B, \mathcal{O}_B(6L))$ such that S is the minimal resolution of the surface $S^* \subset P(2L \oplus 3L \oplus 1)$ defined by the "Weierstrass equation"*

$$y^2 z = x^3 - g_2 x z^2 - g_3 z^3.$$

The only singularities of S^* are rational double points. L is uniquely determined by the projection π. In fact L^{-1} is the normal bundle of any section $\sigma(B)$ in L, and we have deg $(L) = p_g - q + 1$. *The pair (g_2, g_3) are uniquely determined up to the transformation $(g_2, g_3) \mapsto (\lambda^4 g_2, \lambda^6 g_3)$, $\lambda \in \mathbf{C}^*$. The pair (g_2, g_3) satisfy*

(i) $\Delta = 4g_2^3 - 27g_3^2 \not\equiv 0$.

(ii) *For every $t \in B$,* $\min(3 \operatorname{ord}_t (g_2), 2 \operatorname{ord}_t (g_3)) < 12$.

We remark that if S is not a $K3$ surface, then the projection π: $S \to B$ is uniquely determined up to an automorphism of B. In fact, if $q > 0$, the Albanese mapping of S factors through the projection π and the Jacobian mapping of B. If $q = 0$, then the projection π is determined by the linear system $|mK|$, where $m \gg 0$ if S is not a rational surface, and $m \ll 0$ if S is a rational elliptic surface.

Let $Y(n, B) =$ the set of triples (L, g_2, g_3) where L is a line bundle over B with deg $(L) = n$, $g_2 \in H^0(B, \mathcal{O}_B(4L))$, $g_3 \in H^0(B, \mathcal{O}_B(6L))$ and satisfying:

(i) $\Delta = 4g_2^3 - 27g^2 3 \not\equiv 0$.

(ii) For every $t \in B$, $\min(3 \operatorname{ord}_t (g_2), 2 \operatorname{ord}_t (g_3)) < 12$.

Let $X(n, B) = Y(n, B)/\mathbf{C}^*$ where $\mathbf{C}^*$ acts on $Y(n, B)$ by $(L, g_2, g_3) \mapsto (L, \lambda^4 g_2, \lambda^6 g_3)$. There is a natural action of Aut (B) on $X(n, B)$.

THEOREM 2. *There is a* 1-1 *correspondence between elliptic surfaces π: $S \to B$ which admit a section satisfying $p_g - q + 1 = n$, and the set $X(n, B)$. The set of elliptic surfaces S over B (without specifying a projection) which admit a section satisfying $p_g - q + 1 = n$ is in* 1-1 *correspondence with* $X(n, B)/\mathrm{Aut}(B)$, *provid-*

ed that we exclude the case of elliptic K3 surfaces, i.e., $n = 2$, $B = \mathbf{P}^1$.

EXAMPLE. Elliptic surfaces over $\mathbf{P}^1$. There is a unique line bundle L_n of degree n on $\mathbf{P}^1$. We may identify $H^0(\mathbf{P}^1, \mathcal{O}_P,(m))$ with $\mathcal{P}_m$ = the set of polynomials $P(t)$ of degree $\leqslant m$. Then $Y(n, \mathbf{P}^1) = \{(P(t), Q(t)) \in \mathcal{P}_{4n} \times \mathcal{P}_{6n}\}$ satisfying:

(i) $4P(t)^3 - 27Q(t)^2 \not\equiv 0$.

(ii) $\min(3 \operatorname{ord}_a P, 2 \operatorname{ord}_a Q) < 12$ for every $a \in \mathbf{C}$.

(iii) $\min(12n - 3 \deg P, 12n - 2 \deg Q) < 12$.

$\mathbf{C}^*$ acts on $Y(n, \mathbf{P}^1)$ by $(P, Q) \mapsto (\lambda^4 P, \lambda^6 Q)$. $SL(2, \mathbf{C})/\pm 1 = \operatorname{Aut}(\mathbf{P}^1)$ acts on $Y(n, \mathbf{P}^1)$ by

$$(P, Q) \mapsto \left((ct + d)^{4n} P\left(\frac{at + b}{ct + d}\right), (ct + d)^{6n} Q\left(\frac{at + b}{ct + d}\right)\right).$$

REFERENCES

1. E. Brieskorn, *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen*, Math. Ann. **166** (1966), 76–102. MR **34** #6789.

2. F. Hirzebruch, W. D. Neumann and S. S. Koh, *Differentiable manifolds and quadratic forms*, Lecture Notes in Pure and Appl. Math., Dekker, New York, 1971. MR **49** #6250.

3. K. Kodaira, *On compact analytic surfaces*. II, Ann. of Math. (2) **77** (1963), 563–626. MR **32** #1730.

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